

On the mean-width of isotropic convex bodies and their associated L_p -centroid bodies

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Abstract

For any origin-symmetric convex body K in \mathbb{R}^n in isotropic position, we obtain the bound:

$$M^*(K) \leq C\sqrt{n} \log(n)^2 L_K ,$$

where $M^*(K)$ denotes (half) the mean-width of K , L_K is the isotropic constant of K , and $C > 0$ is a universal constant. This improves the previous best-known estimate $M^*(K) \leq Cn^{3/4}L_K$. Up to the power of the $\log(n)$ term and the L_K one, the improved bound is best possible, and implies that the isotropic position is (up to the L_K term) an almost 2-regular M -position. The bound extends to any arbitrary position, depending on a certain weighted average of the eigenvalues of the covariance matrix. Furthermore, the bound applies to the mean-width of L_p -centroid bodies, extending a sharp upper bound of Paouris for $1 \leq p \leq \sqrt{n}$ to an almost-sharp bound for an arbitrary $p \geq \sqrt{n}$. The question of whether it is possible to remove the L_K term from the new bound is essentially equivalent to the Slicing Problem, to within logarithmic factors in n .

1 Introduction

Throughout this work we work in Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. A convex body K in \mathbb{R}^n is a compact convex set with non-empty interior, and the uniform probability measure on K is denoted by λ_K . More generally, it is very useful to consider the larger class of log-concave probability measures μ on \mathbb{R}^n , consisting of absolutely continuous probability measures having density f_μ of the form $\exp(-V)$ with $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ convex. We denote by $\text{Cov}(\mu)$ the covariance matrix of μ , given by $\text{Cov}(\mu) := \int x \otimes x d\mu(x) - \int x d\mu(x) \otimes \int x d\mu(x)$. We will say that μ is isotropic if its barycenter is at the origin and $\text{Cov}(\mu)$ is the identity matrix Id . We will say that a convex body K is isotropic if K has volume one and λ_{K/L_K} is isotropic for an appropriate constant $L_K > 0$, i.e. if its barycenter is at the origin and $\text{Cov}(\lambda_K) = L_K^2 Id$. It is easy to see that by applying an

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affine transformation, any convex body may be brought to isotropic “position”, which is unique up to orthogonal transformations [22]; the isotropic constant L_K is thus an affine invariant associated to any convex body K . See Bourgain [5, 6] and Milman–Pajor [22] for background on the yet unresolved Slicing Problem, which is concerned with obtaining a dimension independent upper-bound on L_K . The current best-known estimate $L_K \leq Cn^{1/4}$ is due to B. Klartag [17], who improved the previous estimate $L_K \leq Cn^{1/4} \log(n)$ of J. Bourgain [6] (see also Klartag–Milman [20] and Vritsiou [31] for subsequent refinements). Throughout this work, all constants c, C, C', \dots denote positive dimension-independent numeric constants, whose value may change from one occurrence to the next. We write $A \simeq B$ to denote that $c \leq A/B \leq C$ for some numeric constants $c, C > 0$.

The L_p -centroid bodies of a given convex body K were introduced by E. Lutwak and G. Zhang in [21] (under different normalization). More generally, given a probability measure μ (having full-dimensional support) and $p \geq 1$, define:

$$h_{Z_p(\mu)}(\theta) = \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p d\mu(x) \right)^{\frac{1}{p}}, \quad \theta \in \mathbb{R}^n.$$

The function $\theta \mapsto h_{Z_p(\mu)}(\theta)$ is a norm on \mathbb{R}^n , and is thus the supporting functional of an origin-symmetric convex body $Z_p(\mu) \subseteq \mathbb{R}^n$ called the L_p -centroid body associated to μ . Note that μ is isotropic iff its barycenter is at the origin and $Z_2(\mu) = B_2^n$, the Euclidean unit-ball. For a log-concave probability measure μ , we also have:

$$1 \leq p \leq q \quad \Rightarrow \quad Z_p(\mu) \subset Z_q(\mu) \subset C \frac{q}{p} Z_p(\mu); \quad (1.1)$$

the first inclusion follows immediately from Jensen’s inequality, and the second is essentially due to Berwald [3] and may be deduced as a consequence of Borell’s lemma [4], see e.g. [22, 25].

The (half) mean-width $M^*(K)$ of a convex body K containing the origin is defined as:

$$M^*(K) := \int_{S^{n-1}} h_K(\theta) d\lambda_{S^{n-1}}(\theta),$$

where $h_K(\theta) = \sup \{ \langle \theta, x \rangle ; x \in K \}$ is the supporting functional of K , S^{n-1} denotes the unit Euclidean sphere and $\lambda_{S^{n-1}}$ denotes the Haar probability measure on S^{n-1} . When K is in addition assumed origin-symmetric, we denote by $\|\cdot\|_K$ the norm on \mathbb{R}^n whose unit-ball is K , and the associated normed space $(\mathbb{R}^n, \|\cdot\|_K)$ is denoted X_K . It was shown by T. Figiel and N. Tomczak-Jaegermann [9] that in this case, there exists a Euclidean structure on \mathbb{R}^n so that $M^*(K)M^*(K^\circ) \leq C \text{Rad}(X_K)$, where K° is the polar body to K , i.e. the unit-ball of the dual norm $\|\cdot\|_K^* = h_K$, and $\text{Rad}(X)$ denotes the norm of the Rademacher projection on $L^2(X)$ (see [28] for more details). Equivalently, we may fix the Euclidean structure and consider linear images (“positions”) of K . A remarkable estimate of G. Pisier [27, 28] asserts that $\text{Rad}(X) \leq C \log(n)$

for all n -dimensional normed spaces, thereby implying the existence of a position of K so that $M^*(K)M^*(K^\circ) \leq C \log(n)$. In particular, since $M^*(K) \geq \text{volrad}(K)$ and $M^*(K^\circ) \geq 1/\text{volrad}(K)$ by the Urysohn and Jensen inequalities, respectively [14], it follows that in the minimal mean-width position of K having unit volume, one has:

$$M^*(K) \leq C\sqrt{n}\text{Rad}(X_K) \leq C'\sqrt{n}\log(n) .$$

Here we denote $\text{volrad}(A) = (\text{Vol}(A)/\text{Vol}(B_2^m))^{1/m}$, the volume-radius of a Borel set $A \subset \mathbb{R}^n$ having m -dimensional linear hull E , with Vol denoting the induced m -dimensional Lebesgue measure on E . An elementary computation verifies that $\text{Vol}(B_2^m)^{1/m} \simeq 1/\sqrt{m}$.

1.1 Mean Width In Isotropic Position

It is nevertheless interesting to check whether other known positions enjoy the same upper-bound on their mean-widths (see e.g. [19, 11, 12] for applications). Our first result asserts that up to the isotropic constant and a logarithmic factor in the dimension, this is indeed the case in the isotropic position:

Theorem 1.1. *Let K denote an origin-symmetric isotropic convex body in \mathbb{R}^n . Then:*

$$M^*(K) \leq C\sqrt{n}\text{Rad}(X_K) \log(1+n)L_K \leq C'\sqrt{n}\log(1+n)^2 L_K .$$

Up to the $\text{Rad}(X_K) \log(1+n)L_K$ term, this bound is best possible, since by Urysohn's inequality $M^*(K) \geq \text{volrad}(K) \simeq \sqrt{n}$. The optimality of the L_K term in this bound is actually intimately connected to the Slicing Problem: removing it would imply a vast improvement over Klartag's best-known bound on the isotropic constant, namely:

$$\begin{aligned} \forall n \geq 1 \quad \forall \text{ isotropic convex } K \subset \mathbb{R}^n \quad M^*(K) &\leq C\sqrt{n}\text{Rad}(X_K) \log(1+n) \Rightarrow \\ \forall n \geq 1 \quad \forall \text{ convex } K \subset \mathbb{R}^n \quad L_K &\leq \inf_{\lambda \in (0,1]} C^{1/\lambda} (\text{Rad}(X_K) \log(1+n))^{1+\lambda} ; \end{aligned}$$

see Proposition 4.2 and the subsequent remark. Note that always $L_K \geq L_{B_2^n} \geq c > 0$ [22]. As for the $\text{Rad}(X_K)$ term, its presence is natural and expected just as in the minimal mean-width position, as easily witnessed by testing $K = \tilde{B}_1^n$, the unit-volume homothetic copy of the unit-ball of ℓ_1^n ; indeed, $M^*(\tilde{B}_1^n) \simeq \sqrt{n}\sqrt{\log(1+n)}$, $L_{\tilde{B}_1^n} \simeq 1$ and $\text{Rad}(\ell_1^n) \simeq \sqrt{\log(1+n)}$ [24]. So some logarithmic dependence in n must ultimately be present and cannot be completely disposed of. The additional $\log(1+n)$ term is probably non-optimal.

The previous best-known upper-bound on the mean-width of an isotropic convex body was $M^*(K) \leq Cn^{3/4}L_K$. This was first shown by M. Hartzoulaki in her Ph.D. Thesis [15], by establishing that the isotropic position is a (one-sided) 1-regular M-position (up to a factor of L_K), and employing Dudley's entropy estimate as in [13] (see below). Other subsequent proofs include that by P. Pivovarov, who employed an approach involving random polytopes [29]. As noticed in [8], this bound is also an

immediate consequence of the following sharp (up to constants) estimate of G. Paouris, valid for an arbitrary isotropic log-concave probability measure μ on \mathbb{R}^n :

$$p \in [1, \sqrt{n}] \Rightarrow M^*(Z_p(\mu)) \leq C\sqrt{p}; \quad (1.2)$$

(in fact, Paouris shows this for all $p \leq q^*(\mu)$, which is equivalent to requiring that the diameter $\text{diam}(Z_p(\mu)) \leq c\sqrt{n}$ for an appropriately small constant $c > 0$, see e.g. [20, Section 4]). Indeed, by (1.1) we have $M^*(Z_n(\mu)) \leq C\frac{n}{\sqrt{n}}M^*(Z_{\sqrt{n}}(\mu)) \leq Cn^{3/4}$. It remains to note that $Z_n(\lambda_K) \simeq \text{conv}(K \cup -K)$ as an easy corollary of the Brunn–Minkowski inequality (e.g. [8]). It immediately follows that for an origin-symmetric isotropic convex body K :

$$M^*(K) \simeq L_K M^*(Z_n(\lambda_{K/L_K})) \leq Cn^{3/4}L_K.$$

Our next result extends Theorem 1.1 to an estimate on $M^*(Z_p(\mu))$ for all $p \geq 1$, thereby extending the estimate (1.2) to the range $p \geq \sqrt{n}$. Inspecting again the example of the uniform measure on $\tilde{B}_1^n/L_{\tilde{B}_1^n}$ illustrates that a logarithmic term must appear in the estimate as p approaches n (either directly or via the norm of the Rademacher projection), and this is indeed the case:

Theorem 1.2. *Let μ denote an isotropic probability measure on \mathbb{R}^n . Then for all $p \geq 1$:*

$$M^*(Z_p(\mu)) \leq C \text{Rad}(X_{Z_p(\mu)}) \max\left(\frac{p \log(1+p)}{\sqrt{n}}, \sqrt{p}\right).$$

As explained above, setting $p = n$ and $\mu = \lambda_{K/L_K}$ in Theorem 1.2 recovers Theorem 1.1. Up to the $\text{Rad}(X_{Z_p(\mu)})$ term, Theorem 1.2 recovers the sharp Paouris bound (1.2) in the range $p \in [1, \sqrt{n}]$. Note that $\text{Rad}(X_{Z_p(\mu)}) \leq C \log(1 + \min(p, n))$, see Section 3. Using in addition (1.1), we summarize the currently best-known estimates:

$$M^*(Z_p(\mu)) \leq C \begin{cases} \sqrt{p} & 1 \leq p \leq \sqrt{n} \\ n^{-1/4}p & \sqrt{n} \leq p \leq \sqrt{n} \log^2(1+n) \\ \sqrt{p} \log(1+n) & \sqrt{n} \log^2(1+n) \leq p \leq n/\log^2(1+n) \\ \frac{p}{\sqrt{n}} \log^2(1+n) & n/\log^2(1+n) \leq p \leq n \end{cases}. \quad (1.3)$$

1.2 Mean Width In Arbitrary Position

In fact, Theorems 1.1 and 1.2 are particular cases of our main result, which we now state in full generality:

Theorem 1.3. *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Let $\lambda_1^2 \geq \dots \geq \lambda_n^2 > 0$ denote the eigenvalues of $\text{Cov}(\mu)$. Then for any $p \geq 1$:*

$$\begin{aligned} M^*(Z_p(\mu)) &\leq C \text{Rad}(X_{Z_p(\mu)}) \frac{1}{\sqrt{n}} \sum_{k=1}^n \max \left(\sqrt{\frac{p}{k}}, \frac{p}{k} \right) (\prod_{i=1}^k \lambda_i)^{\frac{1}{k}} \\ &\simeq C' \text{Rad}(X_{Z_p(\mu)}) \frac{1}{\sqrt{n}} \sum_{i=1}^n \max \left(\sqrt{\frac{p}{i}}, \frac{p}{i} \right) \lambda_i . \end{aligned}$$

Using $\lambda_i \equiv 1$ in the isotropic case, Theorem 1.2 readily follows.

1.3 Covering Estimates

Recall that given two convex bodies K, L in \mathbb{R}^n , the covering number $N(K, L)$ is the minimal number of translates of L whose union covers K . It was shown by Hartzoulaki [15] that an isotropic convex body K in \mathbb{R}^n is (up to the L_K term) in a (one-sided) 1-regular M -position (see [28] for history and terminology), namely:

$$N(K, t\sqrt{n}B_2^n) \leq \exp \left(Cn \frac{L_K}{t} \right) \quad \forall t > 0 . \quad (1.4)$$

We can now improve this for $t \geq C \text{Rad}(X_K)^2 \log^2(1 + \text{Rad}(X_K)) L_K$ by simply invoking Sudakov's inequality (e.g. [28]):

$$N(K, tB_2^n) \leq \exp \left(Cn \frac{M^*(K)^2}{t^2} \right) \quad \forall t > 0 . \quad (1.5)$$

Indeed, coupled with the estimate on $M^*(K)$ from Theorem 1.1, (1.5) immediately implies that an origin-symmetric isotropic convex body K is, up to the $\text{Rad}(X_K) \log(1 + n) L_K$ term, in a (one-sided) 2-regular M -position, namely:

$$N(K, t\sqrt{n}B_2^n) \leq \exp \left(Cn \frac{\text{Rad}(X_K)^2 \log^2(1 + n) L_K^2}{t^2} \right) \quad \forall t > 0 .$$

In fact, one can actually slightly refine this covering estimate as follows:

Theorem 1.4. *For all $t \in [\text{Rad}(X_K) L_K, C\sqrt{n} L_K]$ we have:*

$$N(K, t\sqrt{n}B_2^n) \leq \exp \left(Cn \frac{\text{Rad}(X_K)^2 L_K^2}{t^2} \log^2 \left(1 + \frac{t^2}{\text{Rad}(X_K)^2 L_K^2} \right) \right) .$$

Similar estimates are obtained for L_p -centroid bodies in Section 3.

1.4 Main Ingredient of Proof

We denote by $G_{n,k}$ the Grassmann manifold of all k -dimensional linear subspaces of \mathbb{R}^n ($1 \leq k \leq n$), and given $F \in G_{n,k}$, we denote by P_F the orthogonal projection onto F . Our main result is a rather elementary consequence of the following remarkable theorem of V. Milman and G. Pisier [23], as exposed in [28, Chapter 9], which does not seem to be as well-known as it rightfully should:

Theorem 1.5 (Milman–Pisier).

$$\sqrt{n}M^*(K) \leq C \sum_{k=1}^n \frac{1}{\sqrt{k}} \text{Rad}_k(K) v_k(K) , \quad (1.6)$$

where:

$$v_k(K) := \sup \{ \text{volrad}(P_F K); F \in G_{n,k} \} ,$$

and:

$$\text{Rad}_k(K) := \sup \{ \text{Rad}(X_{P_F K}); F \in G_{n,k} \} .$$

Theorem 1.5 was used in [23] to resolve in the positive a conjecture of R. M. Dudley (see [28]). Indeed, let us compare the estimate (1.6) to Dudley’s entropy estimate:

$$\sqrt{n}M^*(K) \leq C \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} e_k(K) , \quad (1.7)$$

where $e_k(K) := \min \{ t > 0 ; N(K, tB_2^n) \leq 2^k \}$ is the k -th entropy number. By an elementary volumetric estimate, for all $k = 1, \dots, n$:

$$\frac{\text{Vol}(P_F K)}{e_k(K)^k \text{Vol}(P_F B_2^n)} \leq N(P_F K, e_k(K) P_F B_2^n) \leq N(K, e_k(K) B_2^n) \leq 2^k , \quad \forall F \in G_{n,k} ,$$

and therefore $v_k(K) \leq 2e_k(K)$. Consequently, up to the $\text{Rad}_k(K)$ terms, (1.6) should be seen as a (very useful) refinement of (1.7).

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2 Preliminaries

Given $F \in G_{n,k}$, we denote by $\pi_F \mu := \mu \circ P_F^{-1}$ the push-forward of a Borel measure μ on \mathbb{R}^n via P_F . A consequence of the Prekopá–Leindler celebrated extension of the Brunn–Minkowski inequality (e.g. [10]), is that the marginal $\pi_F \mu$ of a log-concave measure μ is itself log-concave on F . This is particularly useful since $P_F Z_p(\mu) = Z_p(\pi_F \mu)$, as follows directly from the definitions.

Recall that the Banach–Mazur distance between two origin-symmetric convex bodies K, L in \mathbb{R}^n is defined as:

$$d_{BM}(K, L) := \inf \left\{ ab ; \frac{1}{b}K \subset T(L) \subset aK , T \in GL(n) \right\} .$$

By John’s Theorem (e.g. [14]), $d_{BM}(K, B_2^n) \leq \sqrt{n}$ for any origin-symmetric convex $K \subset \mathbb{R}^n$.

As for the definition of the Rademacher projection, we refer to [28, 24]. We will only require the following estimate on its norm, due to Pisier (see [28]):

$$Rad(X_K) \leq C \log(1 + d_{BM}(K, B_2^n)) \leq C \log(1 + n) , \quad (2.8)$$

where the second inequality follows by John’s Theorem. In addition, it is easy to show that this norm is self-dual $Rad(X_K) = Rad(X_{K^\circ})$, and since it cannot increase by passing to a subspace, the same holds by duality when passing to a quotient space: $Rad(X_{K \cap E}), Rad(X_{P_E K}) \leq Rad(X_K)$.

The isotropic constant of a log-concave probability measure μ on \mathbb{R}^n having density f_μ is defined as the following affine-invariant quantity:

$$L_\mu := \|f_\mu\|_{L^\infty}^{\frac{1}{n}} (\det \text{Cov}(\mu))^{\frac{1}{2n}} . \quad (2.9)$$

Observe that L_{λ_K} indeed coincides with L_K for a convex body K in \mathbb{R}^n . It was shown by K. Ball [1, 2] that given $n \geq 1$:

$$\sup_{\mu} L_\mu \leq C \sup_K L_K ,$$

where the suprema are taken over all log-concave probability measures μ and convex bodies K in \mathbb{R}^n , respectively (see e.g. [17] for the non-even case).

The fundamental estimate which we employ throughout this work, and which plays an equally fundamental role in previous groundbreaking works of G. Paouris [25, 26] and B. Klartag [18] (see also Klartag–Milman [20]), is given by:

Theorem 2.1 (Paouris, Klartag). *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Then:*

$$\text{volrad}(Z_n(\mu)) \simeq \sqrt{n} \frac{\det \text{Cov}(\mu)^{\frac{1}{2n}}}{L_\mu} \leq C \sqrt{n} \det \text{Cov}(\mu)^{\frac{1}{2n}} .$$

Proof. For the first equivalence, see [26, Proposition 3.7] or [18, Lemma 2.8] in the case that μ is even; in the general case, see [17, Lemma 2.2] and the subsequent computation. The second inequality follows since $L_\mu \geq c > 0$ for any probability measure μ , see e.g. [18]. \square

The following corollary is due to Paouris:

Corollary 2.2 (Paouris). *With the same conditions as above, for any $p \in [1, n]$:*

$$\text{volrad}(Z_p(\mu)) \leq C\sqrt{p} \det \text{Cov}(\mu)^{\frac{1}{2n}}.$$

Proof. As both sides are invariant under volume-preserving linear transformations of \mathbb{R}^n and scale linearly under dilation, we may assume that μ is isotropic. The claim is then the content of [25, Theorem 6.2]. Indeed, we may assume by (1.1) that p is an integer, and so by Alexandrov's inequality between quermassintegrals and Kubota's formula (e.g. [14]), we have:

$$\text{volrad}(Z_p(\mu)) \leq \left(\int_{G_{n,p}} \text{volrad}(P_F Z_p(\mu))^p d\lambda_{G_{n,p}}(F) \right)^{1/p},$$

where $\lambda_{G_{n,p}}$ denotes the Haar probability measure on $G_{n,p}$. Employing Theorem 2.1 for the isotropic log-concave measure $\pi_F \mu$ on $F \in G_{n,p}$, we see that $\text{volrad}(P_F Z_p(\mu)) = \text{volrad}(Z_p(\pi_F \mu)) \leq C\sqrt{p}$, and so the conclusion follows. \square

3 Proofs

Our computations are based on the following immediate corollary of Theorem 2.1 and Corollary 2.2:

Proposition 3.1. *Let μ denote a log-concave probability measure on \mathbb{R}^n with barycenter at the origin. Let $p \geq 1$ and $k = 1, \dots, n$. Then:*

$$v_k(Z_p(\mu)) \leq C\sqrt{\frac{p}{k}} \max(\sqrt{p}, \sqrt{k}) \max_{F \in G_{n,k}} \det \text{Cov}(\pi_F \mu)^{\frac{1}{2k}}.$$

Proof. Let $F \in G_{n,k}$. When $k \leq p$ we use (1.1) and Theorem 2.1:

$$\text{volrad}(P_F(Z_p(\mu))) \leq C\frac{p}{k} \text{volrad}(P_F(Z_k(\mu))) = C\frac{p}{k} \text{volrad}(Z_k(\pi_F \mu)) \leq C'\frac{p}{k} \sqrt{k} \det \text{Cov}(\pi_F \mu)^{\frac{1}{2k}}.$$

When $k \geq p$ we use Corollary 2.2:

$$\text{volrad}(P_F(Z_p(\mu))) = \text{volrad}(Z_p(\pi_F \mu)) \leq C\sqrt{p} \det \text{Cov}(\pi_F \mu)^{\frac{1}{2k}}.$$

Combining the two cases and maximizing over $F \in G_{n,k}$, the assertion immediately follows. \square

Remark 3.2. When μ is the uniform measure on an isotropic convex body, this estimate was already deduced in [11, Theorem 2.4] using a slightly different argument.

3.1 Proof of Theorem 1.3

By the Milman–Pisier Theorem 1.5:

$$\sqrt{n}M^*(Z_p(\mu)) \leq C \sum_{k=1}^n \frac{1}{\sqrt{k}} \text{Rad}_k(Z_p(\mu)) v_k(Z_p(\mu)) . \quad (3.10)$$

Obviously $\text{Rad}_k(Z_p(\mu)) \leq \text{Rad}(X_{Z_p(\mu)})$ by passing to a quotient space. Note that by (1.1) we know that:

$$\frac{1}{C}Z_2(\mu) \subset Z_p(\mu) \subset CpZ_2(\mu) ,$$

for $p \geq 1$, and since $Z_2(\mu)$ is an ellipsoid (the Legendre ellipsoid of inertia), it follows by Pisier’s estimate (2.8) that:

$$\text{Rad}(X_{Z_p(\mu)}) \leq C' \log(1 + d_{BM}(Z_p(\mu), Z_2(\mu))) \leq C' \log(1 + p) .$$

On the other hand, $\text{Rad}_k(L) \leq C \log(1 + k)$ for any origin-symmetric convex L by applying Pisier’s estimate coupled with John’s Theorem (2.8) in dimension k . Consequently:

$$\text{Rad}_k(Z_p(\mu)) \leq \min(\text{Rad}(X_{Z_p(\mu)}), C \log(1 + k)) \leq C' \log(1 + \min(k, p)) .$$

However, as one may check, there will be no loss in the final estimate in using the trivial $\text{Rad}_k(Z_p(\mu)) \leq \text{Rad}(X_{Z_p(\mu)})$.

Now, recalling that $\lambda_1^2 \geq \lambda_2^2 \geq \dots \geq \lambda_n^2 > 0$ denote the eigenvalues of $\text{Cov}(\mu)$, we have by Proposition 3.1:

$$v_k(Z_p(\mu)) \leq C \sqrt{\frac{p}{k}} \max(\sqrt{p}, \sqrt{k}) \max_{F \in G_{n,k}} \det \text{Cov}(\pi_F \mu)^{\frac{1}{2k}} \leq C \sqrt{\frac{p}{k}} \max(\sqrt{p}, \sqrt{k}) (\prod_{i=1}^k \lambda_i)^{1/k} ;$$

(the last inequality is an elementary exercise in linear algebra, for which it may be useful to recall the Cauchy–Binet formula). The first inequality asserted in Theorem 1.3 then immediately follows from (3.10):

$$\sqrt{n}M^*(Z_p(\mu)) \leq C \text{Rad}(Z_p(\mu)) \sum_{k=1}^n \max \left(\sqrt{\frac{p}{k}}, \frac{p}{k} \right) (\prod_{i=1}^k \lambda_i)^{1/k} ;$$

to get a slightly more aesthetically pleasing bound, we may apply the Arithmetic-

Geometric Means Inequality and proceed to estimate:

$$\begin{aligned}
&\leq C \operatorname{Rad}(Z_p(\mu)) \sum_{k=1}^n \max \left(\sqrt{\frac{p}{k}}, \frac{p}{k} \right) \frac{1}{k} \sum_{i=1}^k \lambda_i \\
&= C \operatorname{Rad}(Z_p(\mu)) \sum_{i=1}^n \lambda_i \sum_{k=i}^n \max \left(\sqrt{\frac{p}{k}}, \frac{p}{k} \right) \frac{1}{k} \\
&\leq C' \operatorname{Rad}(Z_p(\mu)) \sum_{i=1}^n \max \left(\sqrt{\frac{p}{i}}, \frac{p}{i} \right) \lambda_i \\
&\leq C' \operatorname{Rad}(Z_p(\mu)) \sum_{i=1}^n \max \left(\sqrt{\frac{p}{i}}, \frac{p}{i} \right) (\Pi_{j=1}^i \lambda_j)^{1/i} ,
\end{aligned}$$

and so the equivalent bound using the arithmetic average follows. The proof of Theorem 1.3 is complete.

3.2 Proof of Theorem 1.4

To show the statement about regularity, we use the following corollary of a slightly more general version of the Milman–Pisier Theorem (see Section 4) coupled with the Pajor–Tomczak-Jaegermann refinement of V. Milman’s low- M^* -estimate, which reads as follows (see the proof of [28, Corollary 9.7]):

$$k^{1/2} c_k(K) \leq C \sum_{j=\lfloor ck \rfloor}^n \frac{1}{\sqrt{j}} \operatorname{Rad}_j(K) v_j(K) \quad \forall k = 1, \dots, n-1 ,$$

where:

$$c_k(K) := \inf \{ \operatorname{diam}(K \cap F) ; F \in G_{n, n-k} \} .$$

Consequently, by Carl’s Theorem [28, Theorem 5.2] and Proposition 3.1 applied to $\mu = \lambda_K$ and $p = n$, we obtain:

$$\begin{aligned}
&\sup_{k=1, \dots, n} \frac{k^{1/2}}{\log(1 + n/k)} e_k(K) \leq C' \sup_{k=1, \dots, n-1} \frac{k^{1/2}}{\log(1 + n/k)} c_k(K) \\
&\leq C'' \sup_{k=1, \dots, n-1} \frac{1}{\log(1 + n/k)} \sum_{j=\lfloor ck \rfloor}^n \frac{1}{\sqrt{j}} \operatorname{Rad}_j(K) v_j(K) \\
&\leq C''' \sup_{k=1, \dots, n-1} \frac{n \operatorname{Rad}(X_K) L_K}{\log(1 + n/k)} \sum_{j=\lfloor ck \rfloor}^n \frac{1}{j} \leq C'''' n \operatorname{Rad}(X_K) L_K .
\end{aligned}$$

In other words:

$$N(K, C\sqrt{n} \operatorname{Rad}(X_K) L_K \sqrt{n/k} \log(1 + n/k) B_2^n) \leq 2^k \quad \forall k = 1, \dots, n .$$

Setting $t = \operatorname{Rad}(X_K) L_K \sqrt{n/k} \log(1 + n/k)$, Theorem 1.4 immediately follows. Note that in isotropic position $K \subset Cn L_K B_2^n$ [22, 16], and so $N(K, t\sqrt{n} B_2^n) = 1$ for $t \geq C\sqrt{n} L_K$.

3.3 Covering L_p -centroid bodies

Similar covering estimates may be deduced for $Z_p(\mu)$. The previous best-known estimate for these covering estimates is due to Giannopoulos–Paouris–Valettas [12], who showed that for any isotropic log-concave measure μ on \mathbb{R}^n and $p \in [1, n]$:

$$N(Z_p(\mu), C_1 t \sqrt{p} B_2^n) \leq \exp \left(C_2 \frac{n}{t^2} + C_3 \frac{\sqrt{n} \sqrt{p}}{t} \right) \quad \forall t \geq 1. \quad (3.11)$$

Note that since $Z_p(\mu) \subset CpZ_2(\mu) = CpB_2^n$ by (1.1), it is enough to only test $t \in [1, \sqrt{p}]$. Also note that setting $\mu = \lambda_K$ and $p = n$, this recovers Hartzoulaki's estimate (1.4).

Invoking Sudakov's inequality (1.5) and using the estimate (1.3) on $M^*(Z_p(\mu))$, an improved covering estimate immediately follows when $t \geq \sqrt{n/p} \log^2(1+n)$. Summarizing the resulting presently best-known estimates when $p \in [1, n/\log^2(1+n)]$ and $t \in [1, \sqrt{p}]$, we have:

$$\log N(Z_p(\mu), C_1 t \sqrt{p} B_2^n) \leq \begin{cases} C_2 \frac{n}{t^2} & 1 \leq t \leq \sqrt{n/p} \\ C_3 \frac{\sqrt{n} \sqrt{p}}{t} & \sqrt{n/p} \leq t \leq \sqrt{n/p} \log^2(1+n) \\ C_4 \frac{n \log^2(1+n)}{t^2} & \sqrt{n/p} \log^2(1+n) \leq t \leq \sqrt{p} \end{cases}.$$

When $n/\log^2(1+n) \leq p \leq n$ one may obtain a slight further improvement beyond Sudakov's inequality, by invoking an argument similar to the one used in the proof of Theorem 1.4; we leave this to the interested reader.

4 Concluding Remarks

4.1 Extended Milman–Pisier Theorem

For completeness, we mention that the Milman–Pisier Theorem 1.5 is in fact a bit more general (see [28, Theorem 9.1]):

Theorem 4.1 (Milman–Pisier). *For any origin-symmetric convex body in \mathbb{R}^n and $j = 0, \dots, n-1$:*

$$\sqrt{n-j} M_{n-j}^*(K) \leq C \sum_{k=\lfloor cj+1 \rfloor}^n \frac{1}{\sqrt{k}} \text{Rad}_k(K) v_k(K),$$

where:

$$M_m^*(K) := \inf \{ M^*(P_F K) ; F \in G_{n,m} \}.$$

Here $M^*(L)$ denotes the (half) mean-width of L in its linear hull $F \in G_{n,m}$, namely:

$$M^*(L) = \int_{S^{n-1} \cap F} h_L(\theta) d\sigma_{S^{n-1} \cap F}(\theta),$$

where $\sigma_{S^{n-1} \cap F}$ denotes the corresponding Haar probability measure. Plugging in the estimates of the previous section, one immediately obtains upper bounds on $M_{n-j}^*(Z_p(\mu))$; we leave this again to the interested reader.

4.2 Removing non-optimal terms

Most probably the $\log(1+n)$ term which appears in our estimates is non-optimal. This is in contrast with the norm of the Rademacher projection term, which should play a role in the estimates (although perhaps with a different power), as in the best-known estimate for the minimal mean-width. To remove the $\log(1+n)$ term, perhaps a majoring-measures type approach in the spirit of Talagrand (see [30]) would be successful. However, this seems difficult at this point.

As for the L_K term, whether it is possible to remove it from our estimate on the mean-width is intimately connected to the Slicing Problem. We refer the reader to the PhD Thesis of K. Ball [1], who showed that when the isotropic constant is bounded then the isotropic position is an M -position, and to the work of Bourgain, Klartag and Milman [7], who conversely showed that if the isotropic position is always an M -position, then the isotropic constant is universally bounded. For completeness, we recall the corresponding arguments:

Proposition 4.2. *Denote:*

$$\begin{aligned} e_m^\wedge &:= \sup \{e_m(K)/\sqrt{m} ; K \text{ is an isotropic convex body in } \mathbb{R}^m\} , \\ L_n &:= \sup \{L_K ; K \text{ is an (isotropic) convex body in } \mathbb{R}^n\} . \end{aligned}$$

Then:

$$L_n \leq \inf_{\lambda \in (0,1]} C^{1/\lambda} (e_{\lfloor (1+\lambda)n \rfloor}^\wedge)^{1+\lambda} .$$

Conversely, for any isotropic convex body K of volume one in \mathbb{R}^n :

$$e_n(K) \leq CL_K \sqrt{n} ,$$

and hence:

$$e_n^\wedge \leq CL_n .$$

Proof. The second assertion follows from the work of Ball [1], who showed that when L_K is bounded, the isotropic position is an M -position. Further refinements pertaining to regularity were obtained by Hartzoulaki (1.4), Giannopoulos–Paouris–Pajor [11] and Giannopoulos–Paouris–Valettas (3.11). Any of these results implies in particular that $e_n(K) \leq C\sqrt{n}L_K$.

To show the first assertion, we use a small variation on the argument from [7]. Let K denote an isotropic convex body in \mathbb{R}^n . Given $m \geq n$, denote by Q_m the following convex body in \mathbb{R}^m :

$$Q_m := \left(\frac{L_{D_{m-n}}}{L_K} \right)^{\frac{m-n}{m}} K \times \left(\frac{L_K}{L_{D_{m-n}}} \right)^{\frac{n}{m}} D_{m-n} ,$$

where D_{m-n} is the homothetic copy of B_2^{m-n} having volume one. It is immediate to verify that Q is isotropic, and consequently $e_m(Q_m) \leq e_m^\wedge \sqrt{m}$. Denoting by E the

subspace spanned by the last $m - n$ coordinates and by B_E its unit Euclidean ball, it is straightforward to verify:

$$N(Q_m \cap E, e_m(Q_m)B_E) \leq N(Q_m, e_m(Q_m)B_2^m) \leq 2^m .$$

On the other hand, a trivial volumetric estimate yields:

$$N(Q_m \cap E, e_m(Q_m)B_E)^{\frac{1}{m-n}} \geq \frac{\text{volrad} \left(\left(\frac{L_K}{L_{D_{m-n}}} \right)^{\frac{n}{m}} D_{m-n} \right)}{\text{volrad}(e_m(Q_m)B_2^{m-n})} \geq \frac{\left(\frac{L_K}{L_{D_{m-n}}} \right)^{\frac{n}{m}} c\sqrt{m-n}}{\sqrt{m}e_m^\wedge} .$$

Combining both estimates and denoting $\lambda = \frac{m-n}{n}$, it follows that:

$$L_K \leq L_{D_{\lambda n}} \left(\frac{1}{c} 2^{\frac{1+\lambda}{\lambda}} \sqrt{\frac{1+\lambda}{\lambda}} \right)^{1+\lambda} \left(e_{n(1+\lambda)}^\wedge \right)^{1+\lambda} .$$

Since $L_{D_m} \simeq 1$ uniformly in m , the first assertion follows. \square

Remark 4.3. Since $e_n(K) \leq CM^*(K)$ by Sudakov's inequality (1.5), it follows that if we could remove the L_K term from our upper bound on $M^*(K)$ given in Theorem 1.1, namely, if:

$$M^*(K) \leq C\sqrt{n}\text{Rad}(X_K)\log(1+n) \leq C'\sqrt{n}\log(1+n)^2 ,$$

for any $n \geq 1$ and origin-symmetric isotropic convex K in \mathbb{R}^n , then we would obtain $L_n \leq \log(1+n)^2 C\sqrt{\log(\log(e+n)^2)}$. In fact, inspecting the proof, we would obtain:

$$L_K \leq \text{Rad}(X_K)\log(1+n)C\sqrt{\log(\text{Rad}(X_K)\log(e+n))} ,$$

since it is easy to verify that $\text{Rad}(X_{Q_m}) \simeq \text{Rad}(X_K)$, uniformly in m .

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